

# Spherical Curvature Inhomogeneities in String Cosmology

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## Abstract

We study the evolution of non-linear spherically symmetric inhomogeneities in string cosmology. Friedmann solutions of different spatial curvature are matched to produce solutions which describe the evolution of non-linear density and curvature inhomogeneities. The evolution of bound and unbound inhomogeneities are studied. The problem of primordial black hole formation is discussed in the string cosmological context and the pattern of evolution is determined in the pre- and post-big-bang phases of evolution.

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## 1 Introduction

Considerable attention has been devoted to exploring the range of behaviours displayed by the equations of string cosmology which are supplied by the variation of the low-energy effective action of the bosonic sector of string theory [1]. Investigations have been made into the evolution of isotropic cosmologies [2], simple homogeneous anisotropic cosmologies of Bianchi type, [3], and Kantowski-Sachs type [4], by various authors, and the present authors have provided a systematic classification of spatially homogeneous string cosmologies in terms of their relative generality when considered as constrained systems of non-linear ordinary differential equations[5].

As in the case of general-relativistic cosmologies, the introduction of inhomogeneities into the string cosmological equations produces a considerable increase in mathematical difficulty: non-linear partial differential equations must now be solved. In practice, this means that we must proceed either by means of approximations which render the non-linearities tractable, or we must introduce particular symmetries into the metric of space-time in order to reduce the number of degrees of freedom which the inhomogeneities can exploit. Accordingly, inhomogeneous string cosmologies have been investigated in the approximation of small perturbations of the isotropic Friedmann-Robertson-Walker (FRW) models, in the 'velocity-dominated' approximation, and by studies of exact inhomogeneous solutions with cylindrical symmetry. Barrow and Kunze [6] found a wide class of exact cylindrically symmetric flat and open inhomogeneous universes. Closed cylindrical solutions were then found by Feinstein et al. [7]. These solutions provide exact 'velocity-dominated'

solutions of general relativity and are expected to form a leading-order approximation to part of the general solution of string cosmology in the neighbourhood of the singularity (if we ignore higher-order string corrections to the action). Veneziano [8] and Buonanno *et al* [9] have studied the behaviour of string cosmologies at early times in the velocity-dominated approximation, (that is, neglecting spatial derivatives, relativistic motions, and 3-curvature inhomogeneities with respect to time-derivatives in the field equations [10]). The asymptotic forms obtained at early times also approximate the behaviour displayed by the exact solutions of Barrow and Kunze on scales larger than the horizon where the inhomogeneities evolve slowly: the inhomogeneities are just homogeneously propagated. However, the exact solutions also provide information about the evolution of dilaton, axion, and gravitational wave inhomogeneities after they enter the horizon, where they attenuate by non-linear oscillations because of the pressure forces exerted by the dilaton and axion fields. These solutions do not contain trapped surfaces and so they cannot be used to follow the collapse of inhomogeneities to black holes, although it would be possible to study this problem by using the closed ( $S^3$ ) solutions studied by Feinstein *et al* in [7].

The exact solutions given in refs. [6] and [7] possess cylindrical symmetry and all physical quantities depend on at most one space coordinate and the time. The case of cylindrical symmetry is natural because of the mathematical simplicity of the field equations whenever there exists a direction in which the pressure equals the energy density. However, it is also important to consider the case where the inhomogeneities possess spherical symmetry. Not only does this seem more natural, in that there need exist no preferred direction in which the inhomogeneity dominates, but it allows the problem of bound inhomogeneities to be addressed more directly without the complication of gravitational wave inhomogeneities.

The choice of spherically symmetric inhomogeneity does not permit exact solutions of Einstein's equations except where fluids have vanishing pressure. However, a clear physical picture of the behaviour of spherically symmetric inhomogeneities with non-zero pressure can be obtained by the device of matching together homogeneous solutions of different curvature and density. The resulting patched solution describes the evolution of spherical overdensities (or underdensities) in a smooth background universe. In the limit that the inhomogeneities become small, they will evolve in accord with the results of small perturbation theory. When the inhomogeneities are not small, we obtain a description of non-linear processes like void formation, the condensation of gravitationally bound lumps, or the creation of primordial black holes. This technique was first introduced into general relativity to study the evolution of inhomogeneities by Lemaître [11] and was subsequently applied to the study of protogalaxies by Harrison [12]. Here, we shall apply it to the equations of string cosmology to further our understanding of the evolution of the pre- and post- big-bang phases in the presence of spherical inhomogeneities.

The string cosmological models considered here are derived from the bosonic sector of heterotic string theory reduced to (3+1) dimensions of spacetime. They are assumed to have vanishing cosmological constant and vanishing Maxwell field. Their field content consists of an antisymmetric tensor field, a dilaton, and the space-time metric tensor. However, in the low-energy limit only the antisymmetric tensor field strength is important in the equations of motion.

In 10-dimensional superstring theory, gravitational anomalies occur which signal the breakdown of energy-momentum conservation. However, by redefining the antisymmetric tensor field strength, these

anomalies can be cancelled. The redefined antisymmetric tensor field strength,  $H$ , is given by [13]

$$H = dB + \omega_L,$$

where  $B$  is the antisymmetric tensor field and  $\omega_L$  is the Lorentz-Chern-Simons (LCS) form involving the Lorentz spin connection. The antisymmetric tensor field strength is a 3-form. In four dimensions it is dual (in the sense of differential forms) to a one-form which can be shown to be the gradient of a scalar field, the axion. FRW spaces are maximally symmetric and so a theorem proved in [14] implies that all contributions from the LCS terms vanish. Hence, the energy-momentum tensor for the axion and the dilaton in the Einstein frame consists of two coupled stiff perfect fluids.

The low-energy limit of string theory provides a new picture for the evolution of the early universe. Once a stage of low coupling and small curvature is reached, the universe enters the 'pre-big-bang' era [15]. During this stage the universe undergoes superinflation (accelerated expansion) in the string frame driven by the kinetic energy of the dilaton. By contrast, in the Einstein frame this corresponds to an accelerated contraction. The pre-big-bang era ends when the string coupling becomes strong enough for the low-energy limit to be no longer valid. However, exactly how this "graceful exit" can be effected is not yet fully understood [16]. In the transition era, complicated non-perturbative effects will become important and it is not clear if a curvature singularity can always be prevented by higher-order contributions. Eventually, the universe must enter the (classical) post-big-bang era. Therefore, the pre-big-bang phase can be understood as a way to provide initial conditions for the classical post-big-bang era. An interesting aspect of this scenario is provided by the duality symmetries present in string theory. In spatially homogeneous cosmological models, scale-factor duality relates solutions for the pre-big-bang phase ("+" branch) to those for the post-big bang phase ("−" branch). Unless there is a self-dual solution there is the problem of how to relate these two branches. This may require an explicitly quantum cosmological transition [17]. In the presence of inhomogeneities, especially those which allow some parts of the Universe to expand whilst other parts collapse, the impact of duality invariance may prove more unusual and motivates further detailed study of realistic inhomogeneous string cosmologies.

Here, we extend our understanding of inhomogeneous string cosmologies by investigating the simple model of non-linear spherically symmetric inhomogeneities outlined above, in which a spherical curvature perturbation is self-modelled by an FRW universe of non-zero curvature in a flat background FRW universe. In section 2 we describe the self-modelling of spherical inhomogeneities in general relativistic universes containing fluids with pressure equal to density. In section 3 the connection with string theory is displayed. In sections 4 and 5 the post and pre-big-bang solutions are given and, finally, the results are discussed in section 6.

## 2 Spherical Inhomogeneities in General Relativity

Spherically symmetric density inhomogeneities can be modelled by matching a section of a closed (or open) FRW universe to a background universe described by a flat FRW universe in such a way that the metric

and its derivatives are continuous at the boundary. First, consider this matching in general relativity for the simple case of a universe containing a perfect fluid, with pressure  $p$  and energy density  $\rho$ , which are related by a stiff equation of state,  $p = \rho$ .

The flat background FRW universe has an expansion scale factor  $R(t)$  and its dynamics are determined by the Friedmann equation

$$\left(\frac{dR}{dt}\right)^2 = \frac{8\pi G}{3}\rho R^2 \quad (1)$$

where  $t$  is proper time. Note that Einstein's equations are invariant under time-reversal, i.e.  $R(t) = R(-t)$ .

Energy-momentum conservation for the  $p = \rho$  fluid implies that

$$\rho \propto R^{-6}. \quad (2)$$

and so  $R(t) \propto t^{\frac{1}{3}}$ .

The dynamics of the perturbed region of non-zero curvature are described by another Friedmann equation, with a scale-factor  $S(\tau)$

$$\left(\frac{dS}{d\tau}\right)^2 = \frac{8\pi G}{3}(\rho + \delta\rho)S^2 - k, \quad (3)$$

where  $\tau$  is the proper time within the density perturbation,  $\delta\rho$ , and the constant,  $k$ , measures its spatial curvature.

The proper time,  $\tau$ , inside the perturbation and that in the background universe,  $t$ , can be related using the equation of relativistic hydrostatic equilibrium [12] [18]

$$\frac{\partial\Phi^{(grav)}}{\partial r} = -\frac{\partial p/\partial r}{p + \rho} \quad (4)$$

where  $\Phi^{(grav)}$  is the (Newtonian) gravitational potential, and  $r$  is the radial distance, and so

$$d\tau = \exp[\Phi^{(grav)}]dt. \quad (5)$$

The equation for hydrostatic equilibrium for a perfect fluid is derived under the assumptions that the configuration is static (that is, only spatial derivatives are non-vanishing) and that the gravitational field is weak, so that the Newtonian gravitational potential  $\Phi^{(grav)}$  completely determines the metric. In this case equation (4) follows directly from the conservation of energy-momentum for a perfect fluid. In this particular matching problem of two Friedmann universes staticity means that there should be no radial flow of matter in or out of the perturbation.

Equations (4) and (5) imply

$$\frac{d\tau}{dt} = \left(\frac{S}{R}\right)^3 [1 + \delta_0]^{-\frac{1}{2}}, \quad (6)$$

where we have introduced the constant  $\delta$ , the density contrast parameter, defined by

$$\delta \equiv \frac{\delta\rho}{\rho}, \quad |\delta| < 1.$$

For the case of an overdensity we have  $\delta > 0$ .

Assume that at some initial time  $t_0$  the perturbation appears and the following matching conditions hold between the metric scale factors inside and outside the perturbation, [19],

$$S_0 = R_0 \quad \left(\frac{dS}{d\tau}\right)_0 = \left(\frac{dR}{dt}\right)_0. \quad (7)$$

Then, we have

$$\left(\frac{dR}{dt}\right)^2 = \frac{\tilde{G}}{R^4}, \quad (8)$$

$$\left(\frac{dS}{d\tau}\right)^2 = \frac{\tilde{G}}{S^4}(\beta - ES^4), \quad (9)$$

where we have defined three new constants by

$$\tilde{G} \equiv \frac{8\pi G}{3}\rho_0 R_0^6, \quad \beta = 1 + \delta_0, \quad E \equiv \frac{\delta_0}{R_0^4}.$$

Furthermore, this implies that  $S$  is given in terms of the background scale-factor,  $R$ , by

$$\frac{dS}{dR} = \frac{S}{R}(1 - FS^4)^{\frac{1}{2}}, \quad (10)$$

where we have defined a further constant by

$$F \equiv \frac{\delta_0}{R_0^4(1 + \delta_0)}. \quad (11)$$

Equation (9) implies that the maximum of the scale-factor of the fluctuation is  $S_+ = F^{-\frac{1}{4}}$ , and hence the integral of (10) is given by

$$S = S_+ \frac{\sqrt{2}(R/R_*)}{\sqrt{1 + (R/R_*)^4}} \quad (12)$$

where  $R_*$  is the value of  $R$  at  $S_+$ . The fluctuation thus begins expanding with the background but is slowed with respect to it because of its overdensity. Eventually, its expansion is halted by its self-gravity and it begins to collapse whilst the background continues to expand.

We see that the spherical fluctuation has vanishing scale-factor  $S$ , i.e. zero radius, for  $R = 0$  and also as  $R \rightarrow \infty$ , where  $S \propto R^{-1}$ . This means that the perturbation does not collapse to a black hole in finite proper time, since  $R \propto t^{\frac{1}{3}}$ . Physically, this situation arises because the  $p = \rho$  fluid possesses a Jeans length equal to the horizon size and the fluid pressure is able to resist gravitational collapse as soon as the fluctuation enters the horizon. This situation is distinctive compared to the evolution of overdensities when  $p < \rho$ . In the  $p < \rho$  case significant overdensities collapse to form black holes if they turn-around when they are intermediate in scale between the particle horizon and the Jeans scale [20]. In practice, even in the  $p = \rho$  case some black-hole formation might be possible in finite time because of small fluctuations in the sound speed and the horizon size but, realistically, we would expect shock formation to play a role in damping non-linear inhomogeneities in the  $p < \rho$  cases (see also the discussions of the scaling properties of black hole formation found by Choptuik and others [21] in this connection).

Our description of the evolution of the spherical overdensity is completed by deriving the ratio of the density in the perturbation to that in the background universe. This is given by

$$\frac{\rho^{(pert)}}{\rho^{(back)}} = (1 + \delta_0) \left( \frac{R}{S} \right)^6.$$

Using equation (12), this can also be expressed as

$$\frac{\rho^{(pert)}}{\rho^{(back)}} = (1 + \delta_0) \left( \frac{R_*}{\sqrt{2}S_+} \right)^6 \left[ 1 + \left( \frac{R}{R_*} \right)^4 \right]^3. \quad (13)$$

If we take the limit of small time (or small  $R$ ) then we recover the usual description of the growth of small perturbations in time in an appropriate gauge.

Finally, it is interesting to note a simple duality scaling property that appears in the above analysis. To see it in context, we can generalise the analysis given above to the case of inhomogeneities in a general perfect fluid model with equation of state  $p = (\gamma - 1)\rho$ . Equation (10) then generalises to

$$\left( \frac{dS}{dR} \right)^2 = \left( \frac{S}{R} \right)^{3\gamma-4} (1 + \delta_0)^{(2-\gamma)/\gamma} \left[ 1 - \left( \frac{S}{S_+} \right)^{3\gamma-2} \right] \quad (14)$$

We see that only in the case of a stiff perfect fluid, that is  $\gamma = 2$ , does (14) admit both a scaling symmetry ( $S \rightarrow \alpha S$ ,  $R \rightarrow \alpha R$ ,  $\alpha$  constant) and the duality invariance

$$R \rightarrow R^{-1}. \quad (15)$$

In Figure 1 we show the evolution of  $R$ ,  $R^{-1}$ , and  $S$  with respect to the background proper time,  $t$ . The background expands as a flat FRW universe ( $R \propto t^{1/3}$ ), whilst the overdensity expands less rapidly, reaches an expansion maximum, but then collapses more slowly, tending to zero size at an infinite future time.

### 3 Connection with String Cosmology

Since FRW universes are spherically symmetric about each point, the LCS contributions to the antisymmetric tensor field strength vanish, and the dilaton and axion fields behave as two coupled stiff ( $\gamma = 2$ ) perfect fluids in the Einstein frame. The Einstein frame is related to the string frame by a conformal transformation of the metric of the form,

$$g_{\mu\nu}^{Einstein} = e^{-\Phi} g_{\mu\nu}^{String}, \quad (16)$$

where  $\Phi$  is the dilaton. The advantage of the Einstein frame is that it is the frame in which the action takes the Einstein-Hilbert form and so, if there are no LCS contributions, the problem is that of general relativity with an energy-momentum tensor consisting of two coupled stiff perfect fluids. However, in order to provide a physical interpretation of the dynamics, we should go back to the string frame. In string theory the equation of motion for a classical test string implies that its world sheet is a minimum surface with respect to the string metric; that is, the (two-dimensional) analog of a geodesic for a point particle. Therefore, it appears that strings “see” the string metric rather than the Einstein metric [22].

The low-energy effective action in the Einstein frame yields to the following set of equations ( $\kappa^2 \equiv 8\pi G, c \equiv 1$ )

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa^2({}^{(\Phi)}T_{\mu\nu} + {}^{(H)}T_{\mu\nu}) \quad (17)$$

$$\nabla_\mu(e^{-2\Phi}H^{\mu\nu\lambda}) = 0 \quad (18)$$

$$\square\Phi + \frac{1}{6}e^{-2\Phi}H_{\mu\nu\lambda}H^{\mu\nu\lambda} = 0 \quad (19)$$

where

$${}^{(\Phi)}T_{\mu\nu} = \frac{1}{2}(\Phi_{,\mu}\Phi_{,\nu} - \frac{1}{2}g_{\mu\nu}(\partial\Phi)^2) \quad (20)$$

$${}^{(H)}T_{\mu\nu} = \frac{1}{12}e^{-2\Phi}(3H_{\mu\lambda\kappa}H_{\nu}{}^{\lambda\kappa} - \frac{1}{2}g_{\mu\nu}H_{\alpha\beta\gamma}H^{\alpha\beta\gamma}) \quad (21)$$

In four dimensions, if we take the space-time dual of the antisymmetric tensor field strength  $H_{\mu\nu\lambda}$ , the Bianchi identity  $dH = 0$  shows that the dynamical content of the antisymmetric tensor field strength is determined by a (pseudo-) scalar field, namely the axion field  $b$ . So, we may write

$$H^{\mu\nu\lambda} = e^{2\Phi}\epsilon^{\mu\nu\lambda\kappa}b_{,\kappa}$$

where  $\epsilon^{\mu\nu\lambda\kappa}$  is the totally antisymmetric Levi-Civita symbol. Furthermore, in the pure dilaton model we will discuss, the perfect fluid is characterised by equal density and pressure,  $p = \rho = \frac{1}{4}\dot{\Phi}^2$ , and the 4-velocity obeys  $u_\alpha u^\alpha = -1$ .

Solutions for flat and closed FRW models have been discussed in [2]. Using conformal time  $\eta$  the metric for a FRW model with a scale factor  $a(\eta)$  can be written as

$$ds^2 = a^2(\eta)(-d\eta^2 + \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2). \quad (22)$$

Naturally, the general-relativistic scenario discussed in the previous section translates directly into a description of the post-big-bang era. However, since the pre-big-bang model makes use of the invariance of solutions under time reversal, it is misleading to make reference to  $S_+$  and  $R_0 = S_0$  since these are two “time-ordered” length scales. If initial conditions are specified at  $R_0 = S_0$  and developed into a state at  $S_+$  then the time reversal causes  $S_+$  to lie before  $R_0 = S_0$ ; that is, conditions at  $R_0$  can no longer be treated as initial conditions. Therefore, in the discussion of pre-big-bang solutions only the initial scale  $R_0 = S_0$  should be used in the solutions for the scale factors.

## 4 Post-Big-Bang Solutions

For simplicity, only overdense perturbations ( $\delta\rho > 0, k > 0$ ) will be discussed in this section. The symbol  $\tau$  denotes conformal time in the flat background universe while  $\tau^{(in)}$  is the conformal time inside the spherical perturbation. Furthermore, the index “\*” refers to the epoch when the scale factor of the perturbation  $S$  has its maximum value,  $S_+$ . Proper time runs from  $0 \rightarrow +\infty$ . Solutions for a pure dilaton model are given in [2].

(i) *Outside the perturbation, modelled as a flat FRW universe:*

The scale factor is given by

$$R^2(\tau) = R_*^2 \frac{\tau}{\tau_*} \quad (23)$$

and the dilaton evolves as

$$e^\Phi = \left(\frac{\tau}{\tau_*}\right)^{\pm\sqrt{3}} \quad (24)$$

(ii) *Inside the perturbation, modelled as a closed FRW universe*

Introducing a function  $\eta$  of the conformal time  $\tau^{(in)}$  inside the perturbation defined by

$$\eta(\tau^{(in)}) = \tan(\tau^{(in)}),$$

the scale factor is given by

$$S^2(\eta) = 2S_+^2 \frac{\eta}{1 + \eta^2} \quad (25)$$

and the dilaton behaves as

$$e^\Phi = \eta^{\pm\sqrt{3}}, \quad (26)$$

since  $\eta_* = 1$ .

In order to relate the conformal times inside and outside the fluctuation, one can make use of equation (12), which gives

$$\frac{\tau}{\tau_*} = \eta. \quad (27)$$

Furthermore, the evolution of the background scale-factor in proper time  $t$ , in the Einstein frame, is found after integrating  $dt = R d\tau$ , to be

$$R = R_* \left( \frac{t}{t_*} \right)^{\frac{1}{3}} \quad (28)$$

As discussed above, the frame appropriate for physical interpretation is the string frame. In this frame, the background scale factor is given by

$${}^{(S)}R = e^{\Phi/2} R = R_* \left( \frac{\tau}{\tau_*} \right)^{\frac{1}{2}(1 \pm \sqrt{3})}. \quad (29)$$

Expressed in proper time in the string frame, i.e. after integrating  $d^{(S)}t = {}^{(S)}R d\tau$ ,  ${}^{(S)}R({}^{(S)}t)$  is given by

$${}^{(S)}R = R_* \left( \frac{{}^{(S)}t}{{}^{(S)}t_*} \right)^{\pm \frac{1}{\sqrt{3}}}. \quad (30)$$

Pre- and post-big-bang, respectively, refer to the background universe. In the post-big-bang epoch the upper sign in (30) is chosen. The matching conditions (cf (7)) provide a well-behaved metric in the Einstein frame. Since physics should not be frame dependent, it is expected that similar matching conditions should hold in the string frame. As one can show, this implies that the sign chosen in (26) for the evolution of the dilaton inside the perturbation should be the same as that in the background universe; hence

$$\exp\left(\Phi^{(back)}\right) = \exp\left(\Phi^{(pert)}\right). \quad (31)$$

The energy densities in the Einstein frame and the string frame can be related by going back to the definition of the energy-momentum tensor as a variational derivative of the Lagrangian  $\mathcal{L}$  [23]. In the Einstein frame, we have

$$\rho = g^{00} T_{00} = (-g)^{-\frac{1}{2}} g^{00} \frac{\delta \mathcal{L}}{\delta g^{00}}.$$

Using the conformal transformation given by equation (16), we find that the energy density in the string frame is given by

$$^{(S)}\rho = e^{-2\Phi}\rho. \quad (32)$$

Hence, using (31), we confirm that the ratio of the energy density in the perturbation to that in the background is the same in the Einstein and the string frames:

$$\frac{^{(S)}\rho^{(pert)}}{^{(S)}\rho^{(back)}} = \frac{\rho^{(pert)}}{\rho^{(back)}}. \quad (33)$$

## 5 Pre-Big-Bang Solutions

In order to find solutions depending on one length scale we begin again with equation (10). Furthermore, since the analysis does not depend on a maximal length scale, the solutions for underdense regions (modelled by matching to an open FRW universe, with  $k < 0$ ), can be treated as well.

### 5.1 Underdense Regions

In this case the density parameter  $\delta$  is negative and hence the constant  $F$ , defined in (11), is also negative. Equation (10) now becomes

$$\frac{dS}{dR} = \frac{S}{R} \left( 1 + |F| S^4 \right)^{\frac{1}{2}}. \quad (34)$$

This integrates to give

$$\frac{S}{S_0} = \frac{\sqrt{2} \left[ (1 + \delta_0)^{-\frac{1}{2}} - 1 \right]^{\frac{1}{2}} \left( \frac{R}{R_0} \right)}{\left[ \frac{|\delta_0|}{1 + \delta_0} - \left[ (1 + \delta_0)^{-\frac{1}{2}} - 1 \right]^2 \left( \frac{R}{R_0} \right)^4 \right]^{\frac{1}{2}}}. \quad (35)$$

Hence, the ratio of the density in the fluctuation to that in the background (in the Einstein frame) is given by

$$\frac{\rho^{(pert)}}{\rho^{(back)}} = \frac{1 + \delta_0}{8 \left[ (1 + \delta_0)^{-\frac{1}{2}} - 1 \right]^3} \left[ \frac{|\delta_0|}{1 + \delta_0} - \left[ (1 + \delta_0)^{-\frac{1}{2}} - 1 \right]^2 \left( \frac{R}{R_0} \right)^4 \right]^3. \quad (36)$$

## 5.2 Overdense Regions

In this case both the density parameter,  $\delta$ , and the constant  $F$  are positive. An integration of equation (10) yields

$$\frac{S}{S_0} = \frac{\sqrt{2} \left[ 1 - (1 + \delta_0)^{-\frac{1}{2}} \right]^{\frac{1}{2}} \left( \frac{R}{R_0} \right)}{\left[ \frac{\delta_0}{1 + \delta_0} + \left[ 1 - (1 + \delta_0)^{-\frac{1}{2}} \right]^2 \left( \frac{R}{R_0} \right)^4 \right]^{\frac{1}{2}}}, \quad (37)$$

and the ratio of the densities in the Einstein frame is given by

$$\frac{\rho^{(pert)}}{\rho^{(back)}} = \frac{1 + \delta_0}{8 \left[ 1 - (1 + \delta_0)^{-\frac{1}{2}} \right]^3} \left[ \frac{\delta_0}{1 + \delta_0} + \left[ 1 - (1 + \delta_0)^{-\frac{1}{2}} \right]^2 \left( \frac{R}{R_0} \right)^4 \right]^3. \quad (38)$$

In order to connect the conformal times inside and outside the perturbation, and hence the solutions for the pure dilaton model, we will use the original notation of [2].

(i) *In the flat background* the scale factor is given by [2]

$$R^2 = \frac{K}{\sqrt{3}} \tau \quad (39)$$

with  $K$  some constant and  $\tau$  the conformal time in the background. The evolution of the dilaton is given by

$$e^\Phi = \left( \frac{\tau}{\tau_f} \right)^{\pm \sqrt{3}} \quad (40)$$

where  $\tau_f$  is an integration constant. For pre-big-bang solutions the  $-$  sign is chosen. In terms of proper time in the Einstein frame these solutions read

$$e^\Phi = \left( \frac{t}{t_f} \right)^{-\frac{2}{\sqrt{3}}} \quad (41)$$

and

$$R^2 = \left( \frac{3}{2} \frac{K}{\sqrt{3}} t \right)^{\frac{2}{3}}. \quad (42)$$

Recalling that time is negative in the pre-big-bang era, one sees that in the Einstein frame the background universe is contracting and the dilaton is described by a growing function. The scale factor in the string frame is given by

$${}^{(S)}R^2 = e^\Phi R^2 = \left(\frac{3}{2}\right)^{\frac{2}{3}} t_f^{\frac{2}{\sqrt{3}}} \left(\frac{K}{\sqrt{3}}\right)^{\frac{2}{3}} t^{\frac{2}{3}(1-\sqrt{3})}, \quad (43)$$

and hence the universe is expanding in the string frame.

(ii) *Inside the density fluctuation* the scale factor is given by

$$S^2 = \frac{\tilde{K}}{\sqrt{3}} \frac{\eta}{(1 + k\eta^2)}, \quad (44)$$

with  $\tilde{K}$  some constant, and

$$\eta = \begin{cases} \tan(\tau_{in}), & k = +1 \quad \text{overdensity} \\ \tanh(\tau_{in}), & k = -1 \quad \text{underdensity} \end{cases}$$

where  $\tau_{in}$  is the conformal time inside the fluctuation. Using the matching conditions, (7),  $\tau_0$  and  $\eta_0$  can be related at this epoch by

$$\tau_0 = \frac{\eta_0}{1 - k\eta_0^2}. \quad (45)$$

The solution for the dilaton is found to be

$$e^\Phi = \left(\frac{\eta}{\eta_f}\right)^{-\sqrt{3}}, \quad (46)$$

with  $\eta_f$  an integration constant.

Since the dilaton solutions are involved in the transformation of the energy densities from the Einstein frame to the string frame, it is enough to relate  $\tau$  and  $\eta$  (and not explicitly the conformal times inside and outside the fluctuation) and this can be done using equations (35) and (37). First, note that

$$\left(\frac{R}{R_0}\right)^2 = \frac{\tau}{\tau_0} \quad (47)$$

and

$$\left(\frac{S}{S_0}\right)^2 = \frac{\eta(1 + k\eta_0^2)}{\eta_0(1 + k\eta^2)}. \quad (48)$$

Observing that  $\delta_0$  is positive or negative, according to the sign of  $k$ , equations (35) and (37) can be reduced to a single expression, namely

$$\left(\frac{S}{S_0}\right)^2 = \frac{2\alpha \left(\frac{R}{R_0}\right)^2}{\gamma + \alpha^2 \left(\frac{R}{R_0}\right)^4} \equiv \Gamma\left(\frac{\tau}{\tau_0}\right) \quad (49)$$

with

$$\alpha \equiv 1 - (1 + \delta_0)^{-\frac{1}{2}} \quad , \quad \gamma \equiv \frac{\delta_0}{1 + \delta_0}.$$

The function  $\Gamma(\frac{\tau}{\tau_0})$  is defined in accord with equation (47). So, using equation (48), the function  $\eta$  can be expressed in terms of the conformal time  $\tau$  in the background universe,

$$\eta = k \frac{1 - \sqrt{1 - 4kE_0^2\Gamma^2}}{2E_0\Gamma}, \quad (50)$$

where

$$E_0 \equiv \frac{\eta_0}{1 + k\eta_0^2}$$

which can be expressed in terms of  $\tau_0$  using (45).

Therefore, in this more general case, the relationship between the dilaton solutions inside and outside the fluctuation is more complicated than that found earlier for the post-big-bang era (cf equation (31)).

The perturbation originates at some epoch  $\tau_0$  with a scale factor corresponding to a value  $R_0$  in the background universe. The big-bang occurs at  $\tau = 0$ , but one should not follow the evolution all the way to  $\tau = 0$  since at some point the string coupling becomes too strong for the low-energy action to remain a valid approximation to the full theory. In the pre-big-bang phase the proper time is restricted to  $t < 0$ . Suppose that a perturbation originates at  $t_0$  and the period of interest is between  $t_0$  and some time  $t_s$ , (with  $|t_s| < |t_0|$ ), when the string coupling becomes strong, so we are interested in the regime  $\tau < \tau_0$ ,  $R < R_0$ . Assuming  $R \ll R_0$ , the scale factor of the perturbation  $S$  and the function of its conformal time  $\eta(\tau)$  can be expanded as follows,

$$\frac{S}{S_0} \simeq \left(\frac{2\alpha}{\gamma}\right)^{\frac{1}{2}} \frac{R}{R_0} \quad (51)$$

$$\frac{\eta}{\eta_0} \simeq \frac{2\alpha}{\gamma} (1 + k\eta_0^2)^{-1} \frac{\tau}{\tau_0}. \quad (52)$$

Thus,  $\frac{S}{S_0} \sim \frac{R}{R_0}$ ,  $\frac{\eta}{\eta_0} \sim \frac{\tau}{\tau_0}$ . However, for  $\delta_0 > 0$ , the proportionality factor in (51) is bigger than unity, and so the scale factor of the perturbation exceeds that of the background universe (see figure 2).

This might be interpreted as interchanging the role of the flat and the closed FRW models (*ie* there are inhomogeneities corresponding to flat space-time sections in a closed FRW background universe [24]).

Figures 3 and 4 show the behaviour of the two scale factors,  $R$  and  $S$ , with time,  $t$ , for different choices of the initial density parameter  $\delta_0$ . In Figure 3, the scale factor of the overdense region is larger than that of the background universe within the period of interest between  $t_0$  up to some time,  $t_s$ . In Figure 4, where there is an initial underdensity at  $t_0$ , the scale factor of the perturbation stays below that of the background universe.

It is clear that the evolution of the two scale factors is very similar, from the origin of the perturbation up to the big bang. Thus, one cannot actually say that the perturbation evolves independently from the background. Qualitatively, the same behaviour is found in the string frame. The evolution of the dilaton depends on the respective conformal times (or functions of them), namely  $\tau$  and  $\eta$ . However, in the regime between the origin of the perturbation and the big bang these are directly proportional to each other as can be seen from (52). Hence, the evolution of the scale factors in the string frame is very similar as well.

In accord with the behaviour in the general relativistic (hence post-big bang) regime, no primordial black holes should be formed in the pre-big-bang era in this model since the perturbation does not evolve independently and so does not form trapped surfaces during the collapse of the scale factor of the background universe in the Einstein frame.

From this discussion it can be concluded that the introduction of a spherical perturbation modelled as a closed or open FRW model in a flat FRW model does not destroy the global isotropy of the flat background universe. Hence the pre-big-bang flat FRW solution is robust with respect to this special type of curvature perturbation.

In heterotic string theory, any solutions containing only a dilaton can be transformed into solutions containing a dilaton and axion without changing the background metric. S-duality reflects the fact that heterotic string theory is invariant under  $SL(2, IR)$  transformations. Define a complex scalar  $\lambda = b + ie^{-\Phi}$  then [25]

$$\lambda \rightarrow \lambda' = \frac{\alpha\lambda + \beta}{\gamma\lambda + \delta} \quad g_{\mu\nu} \rightarrow g_{\mu\nu} \quad \alpha, \beta, \gamma, \delta \in IR, \quad \alpha\delta - \beta\gamma = 1$$

is a solution to the low-energy equations of motion (equations (17)-(19)).

If we start with a pure dilaton solution

$$\lambda = ie^{-\Phi}$$

and choose  $\alpha = \beta = \gamma = -\delta = 1/\sqrt{2}$ , then  $\lambda' = b^{new} + i \exp[-\Phi^{new}]$  is given by

$$b^{new} = \tanh \Phi \tag{53}$$

$$e^{\Phi^{new}} = \cosh \Phi. \tag{54}$$

Hence, the pure FRW dilaton solution has

$$e^{\Phi} = \left( \frac{\tau}{\tau_0} \right)^{\pm\sqrt{3}},$$

where  $\tau$  is a different function of the conformal time  $\eta$  for each of the FRW models which can be read of from the solutions given above, and is related to a FRW dilaton-axion solution with

$$e^\Phi = \frac{1}{2} \left[ \left( \frac{\tau}{\tau_0} \right)^{\sqrt{3}} + \left( \frac{\tau}{\tau_0} \right)^{-\sqrt{3}} \right], \quad (55)$$

$$b = \pm \frac{\left( \frac{\tau}{\tau_0} \right)^{\sqrt{3}} - \left( \frac{\tau}{\tau_0} \right)^{-\sqrt{3}}}{\left( \frac{\tau}{\tau_0} \right)^{\sqrt{3}} + \left( \frac{\tau}{\tau_0} \right)^{-\sqrt{3}}}, \quad (56)$$

which is given in [2].

However, near the big-bang (that is, for small  $\frac{\tau}{\tau_0}$ ), the dilaton solution goes over into that obtained for the pure dilaton case and so the conclusions from the pure dilaton case are not likely to be changed fundamentally.

## 6 Conclusions

A very simple inhomogeneous string cosmological model has been investigated in the pre- and post-big-bang eras. In the post-big-bang era the usual general relativistic behaviours of a growing overdensity or underdensity are found. This leads to an increasingly inhomogeneous universe if the initial inhomogeneities are significant, or they are not inflated away. However, in the pre-big-bang phase the global isotropy of the background FRW model is unaffected by the introduction of a spherically symmetric curvature perturbation. This rather different behaviour of the model in the pre- and post-big-bang stages is a manifestation of the fact that these two regimes pick out different parts of the general relativistic solution. In general relativity, considering only positive (proper) times, the solution is naturally divided by the time  $t_0$  when the perturbation originates. The physical solution is then given for times later than  $t_0$ , whereas the part between the big bang and  $t_0$  is discarded. However, in the pre-big-bang stage one is considering a time-reflection of the general relativistic solution. Hence, the part of the solution that was discarded in the general relativistic case (post-big-bang) becomes the physical solution in the pre-big-bang era. Thus, it is the physical solution of general relativity which is considered unphysical in the pre-big-bang regime. Hence, unlike in the general relativistic solution, a curvature perturbation like that considered here does not lead to an inhomogeneization of the universe in the pre-big-bang epoch.

The authors of references [8] and [9] discussed inhomogeneous pre-big-bang models using standard approximation techniques of general relativistic cosmology. They found that the approximation of neglecting spatial gradients becomes better as the big-bang singularity is approached. This conclusion is confirmed by the simple quasi-homogeneous model discussed here. The evolution of a curvature perturbation modelled by a section of a Friedmann universe is very similar to that of the background Friedmann universe and indeed becomes more similar to it with time. Hence isotropy is conserved and spatial gradients do not become important. We note also that part of the general solution for general relativistic cosmological models

containing a  $p = \rho$  perfect fluid is a perturbation of a Kasnerlike solution which (unlike for case of perfect fluids with equation of state  $p < \rho$ ) contains the isotropic expansion as a particular case. Because of this, the general behaviour of the equations of string cosmology at early times is significantly simpler than that of general relativity in vacuum.

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## References

- [1] An archive for string cosmology maintained by M. Gasperini can be found at <http://www.to.infn.it/teorici/gasperini/>
- [2] E. J. Copeland, A. Lahiri, D. Wands, Phys. Rev. D **50**, 4868 (1994).
- [3] M. Gasperini, R. Ricci, Class. Quant. Grav. **12**, 677 (1995); N.A. Batakis, Phys. Lett. B **353**, 39 (1995); N.A. Batakis, A.A Kehagias, Nucl. Phys. B **449**, 248 (1995).
- [4] J.D. Barrow and M. Dabrowski, Phys. Rev. D **55**, 630 (1997).
- [5] J.D. Barrow and K.E. Kunze, Phys. Rev. D **55**, 623 (1997).
- [6] J.D. Barrow and K.E. Kunze, Phys. Rev. D **56**, 741 (1997).
- [7] A. Feinstein, R. Lazkoz, and M.A. Vázquez-Mozo, Phys. Rev. D hep-th/9704173.
- [8] G. Veneziano, “Inhomogeneous Pre-Big Bang String Cosmology” hep-th/9703150, Phys. Lett. B
- [9] A. Buonanno, K. A. Meissner, C. Ungarelli, G. Veneziano “Classical Inhomogeneities in String Cosmology” hep-th/9706221.
- [10] The ‘velocity-dominated’ approximation, originally introduced in a formal way by D. Eardley, R. Sachs and E.P.T. Liang, J. Math. Phys. **13**, 99 (1972) and E.P.T. Liang, J. Math. Phys. **13**, 386 (1972), following its use by Landau and Lifshitz in *The Classical Theory of Fields*, Pergamon, Oxford (1962) and E.M. Lifshitz and I. Khalatnikov, Adv. Phys. **12**, 208 (1963). It was also introduced by K. Tomita, Prog. Theo. Phys. **48**, 1503 (1972) under the name ‘anti-Newtonian approximation’, so called because neglecting space derivatives ( $\partial/\partial x$ ) with respect to time derivatives ( $\partial/\partial(ct)$ ) is equivalent to taking a limit in which the speed of light goes to zero ( $c \rightarrow 0$ ) — the antithesis of Newtonian physics in which  $c = \infty$ .
- [11] G. Lemaître, Ann. Soc. Sci. Bruxelles A **53**, 51 (1933).

- [12] E.R. Harrison Phys. Rev. D **1**, 2726 (1970); E.R. Harrison in *Cargèse Lectures in Physics*, Vol. 6, ed E. Schatzmann, Gordon & Breach, NY, (1973).
- [13] M. B. Green, and J. H. Schwarz, Phys. Lett. **149B** 117 (1984); M. B. Green, J. H. Schwarz, and E. Witten *Superstring Theory* Volume 2 CUP (1987).
- [14] B. A. Campbell, M. J. Duncan, N. Kaloper, and K. A. Olive, Nucl. Phys. B **351** 778 (1991).
- [15] G. Veneziano, Phys. Lett. B **265** 287 (1991); M. Gasperini, and G. Veneziano, Astropart. Phys. **1** 317 (1993); see also <http://www.to.infn.it/teorici/gasperini/>
- [16] M. Gasperini, M. Maggiore, G. Veneziano “Towards a non-singular pre-big-bang cosmology” hep-th/9611039; R. Brustein, R. Madden “Graceful exit and energy conditions in string cosmology” hep-th/9702043.
- [17] M. Gasperini, G. Veneziano, Gen. Rel. Grav. **28**, 1301 (1996); M. Dabrowski, C. Kiefer, Phys. Lett. B **397**, 185 (1997).
- [18] L. D. Landau, and E. M. Lifshitz *Classical Theory of Fields*, 4th edn., Pergamon, Oxford (1975).
- [19] For a summary of formal matching conditions in general relativity see H. Stephani, *General Relativity*, CUP, Cambridge, (1982), section 16.5.
- [20] B.J. Carr and S.W. Hawking, MNRAS **168**, 399 (1974), B.J. Carr, Astrophys. J. **201**, 1 (1975).
- [21] M.W. Choptuik, Phys. Rev. Lett. **70**, 9 (1993); J.C. Niemeyer and K. Jedamzik, astro-ph/9709072.
- [22] J. A. Harvey, A. Strominger “Quantum Aspects of Black Holes” 1992 Trieste Spring School on String Theory and Quantum Gravity; 1992 TASI Summer School in Boulder, Colorado; hep-th/9209055
- [23] S. Kalara, N. Kaloper, and K. A. Olive Nucl. Phys. B **341** 252 (1990).
- [24] The global behaviour of such inhomogeneities and their relation to the problem of when closed universes (ie those with compact spatial hypersurfaces) recollapse is discussed by Y. B. Zeldovich and L. Grishchuk, MNRAS **207**, 23P (1984), J.D. Barrow and F.J. Tipler, MNRAS **216**, 395 (1985) and J.D. Barrow, G. Galloway, and F.J. Tipler, MNRAS **223**, 835 (1986).
- [25] A. Sen, Int. J. Mod. Phys. A **9** 3707 (1994).

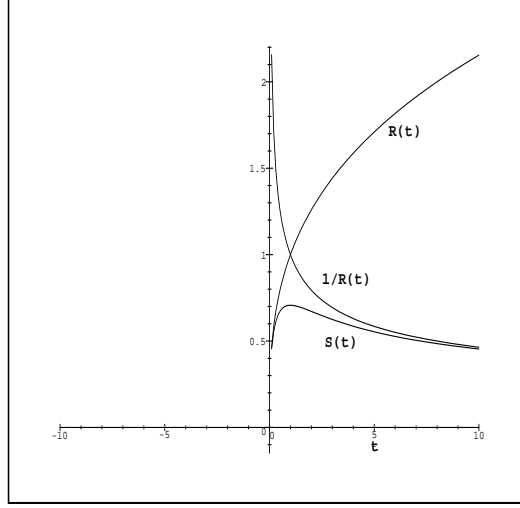


Figure 1: Evolution of the background scale factor  $R \propto t^{\frac{1}{3}}$ , the transformed one  $R^{-1} \propto t^{-\frac{1}{3}}$  and the scale factor of the perturbation  $S(t) \propto t^{\frac{1}{3}}(1 + t^{\frac{4}{3}})^{-\frac{1}{2}}$  in terms of proper background time  $t$ .

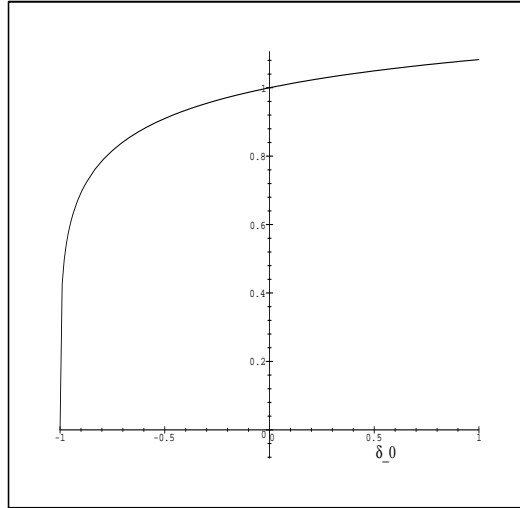


Figure 2: The proportionality factor  $\left(2\frac{\alpha}{\gamma}\right)^{\frac{1}{2}}$  as a function of  $\delta_0$ .

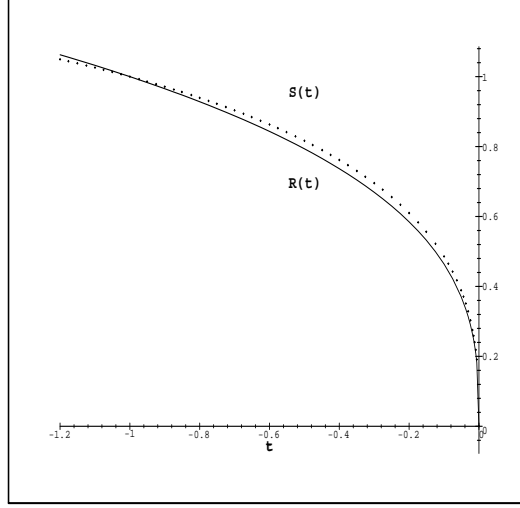


Figure 3: Evolution of the scale factors of the perturbation  $S$  and the background universe  $R$  as a function of proper time  $t$  in the Einstein frame ;  $\delta_0 = 0.5$ ,  $R_0 = 1$ ,  $t_0 = -1$ .

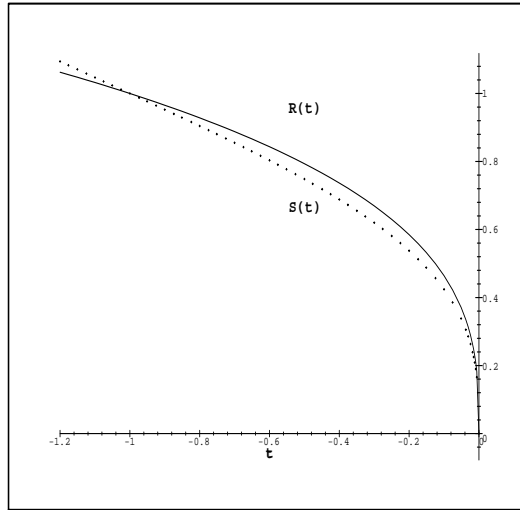


Figure 4: Evolution of the scale factors of the perturbation  $S$  and the background universe  $R$  as a function of proper time  $t$  in the Einstein frame;  $\delta_0 = -0.5$ ,  $R_0 = 1$ ,  $t_0 = -1$ .